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On the Zeros of Incomplete Polynomials

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In this paper we will consider "incomplete" polynomials of the form

$$P(z) = z^n + a_{n-k-1}z^{n-k-1} + a_{n-k-2}z^{n-k-2} + \cdots + a_0, \tag{1}$$

where $k > 0$ and $|a_0| + |a_1| + \cdots + |a_{n-k-1}| \neq 0$, and investigate the number of zeros of $P(z)$ falling within an angle whose vertex is at the origin of the complex plane.

$n - k$ zeros of such a polynomial can be found in every angle (even a ray) whose vertex is at the point $z = 0$. In fact, let $z_1, z_2, \ldots, z_{n-k}$ be positive numbers, and let $Q(z)$ be a polynomial of degree $n - k - 1$ which interpolates the function $-z^n$ at the points $z_1, z_2, \ldots, z_{n-k}$:

$$Q(z_s) = -z_s^n, \qquad s = 1, 2, \ldots, n - k.$$

Then $z_1, z_2, \ldots, z_{n-k}$ are zeros of the polynomial

$$z^n + Q(z),$$

which is of the form (1).

On the other hand, it is not hard to see that, in the case $k = 1$, all zeros of the polynomial (1) (in this case

$$P(z) = z^n + a_{n-2}z^{n-2} + \cdots + a_0)$$

cannot be situated on the same side of any straight line which passes through $z = 0$, because the sum of the zeros of this polynomial is zero.

In the general case, $k \geqslant 0$, we have the following:

THEOREM. *Every polynomial of the form*

$$P(z) = z^n + a_{n-k-1}z^{n-k-1} + a_{n-k-2}z^{n-k-2} + \cdots + a_0,$$

where

$$n - 1 \geqslant k \geqslant 0, \qquad |a_0| + |a_1| + \cdots + |a_{n-k-1}| \neq 0$$

can have no more than $n - k$ zeros falling within an angle U, less than $2\pi/(k + 1)$, whose vertex is at the origin of the complex plane.

If U is equal to $2\pi/(k+1), k > 1$, this polynomial can have $n-k+1$ zeros in this angle only when all these zeros are on the sides of the angle, and

$$P(z) = z^n + a_{n-k-1}z^{n-k-1}, \qquad a_{n-k-1} \neq 0.$$

As a direct corollary of this theorem, we have that for an angle U less than $2\pi/(k-s)$, where $s = -1, 0, 1,..., k-1$, a polynomial of the form (1) can have no more than $n-k+s+1$ zeros within this angle.

By substituting $z = 1/y$, we can conclude similar results for incomplete polynomials of the form

$$a_0 y^n + a_1 y^{n-1} + \cdots + a_{n-k-1} y^{k+1} + 1.$$

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In order to prove the theorem, it is enough to consider the case where U is of the following form:

$$U\colon 0 \leqslant \arg z \leqslant \varphi,\ \varphi \leqslant 2\pi/(k+1). \tag{2}$$

The general case is then obtained by rotation of the coordinate system about the origin.

To facilitate proof of the theorem, we will derive first some lemmas.

LEMMA 1. *Let $[z_1, z_2, ..., z_{n-k+1}]^k$ denote the sum*

$$\sum_{i_1+i_2+\cdots+i_{n-k+1}=k} z_1^{i_1} z_2^{i_2} \cdots z_{n-k+1}^{i_{n-k+1}}. \tag{3}$$

If $z_1, z_2, ..., z_{n-k+1}$ are the zeros of a polynomial of the form (1) *then*

$$[z_1, z_2, ..., z_{n-k+1}]^k = 0.$$

Proof. Since z_i for $i = 1, 2,..., n-k+1$ are zeros of the polynomial of form (1) then the polynomial

$$P_{n-k-1}(z) = a_{n-k-1}z^{n-k-1} + a_{n-k-2}z^{n-k-2} + \cdots + a_0$$

interpolates the function $-z^n$ at the points z_i and according to Newton's formula [1]:

$$\begin{aligned} P_{n-k-1}(z) = c_0 + c_1(z-z_1) + c_2(z-z_1)(z-z_2) + \cdots \\ + c_{n-k}(z-z_1)(z-z_2)\cdots(z-z_{n-k}). \end{aligned} \tag{4}$$

It is known [1] that

$$c_j = [z_1, z_2, \dots, z_{j+1}]^{n-j}$$

and thus from (4) we get

$$c_{n-k} = [z_1, z_2, \dots, z_{n-k+1}]^k = 0. \tag{5}$$

[Equation (5) is a sufficient condition for $z_1, z_2, \dots, z_{n-k+1}$ to be zeros of some polynomial of form (1).]

This lemma is a special case of the more general result due to E. Kiryatski [2]. From Lemma 1, as in [3], we can see that in the angle $0 \leqslant \arg z < \pi/k$ the polynomial (1) can not have more than $n - k$ zeros. In fact, in this case, each term in the sum (3) has a positive imaginary part.

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LEMMA 2. *Let U be the smallest angle, whose vertex is at the origin of the complex plane, which contains all the points $y_1, y_2, \dots, y_N$, where $y_i \neq 0$ for $i = 1, 2, \dots, N$; $N \geqslant 2$ and*

$$[y_1, y_2, \dots, y_N]^k = 0. \tag{6}$$

Then in this angle U, there are the points $y_1', y_2', \dots, y_s'$; $y_i' \neq 0$, $i = 1, 2, \dots, s$; $N \geqslant s \geqslant 1$, such that

$$[y_1', y_2', \dots, y_s']^k = 0 \tag{7}$$

and all of these points are situated on two rays coming out of the origin.

Proof. Let U be the smallest angle whose vertex is on the origin of the complex plane, which contains all the points $y_1, y_2, \dots, y_N$. Let q of these points be within U (not on the sides). If $q = 0$ the lemma is true. Let us assume that for $q \leqslant p$ the lemma is also true. Now let $q = p + 1$ and the point y_1 is within U. Then only p of the points $y_2, y_3, \dots, y_N$ fall within U.

Now let us examine the equation $L(t, z) = 0$, where

$$L(t, z) = [z, y_2, ty_3, ty_4, \dots, ty_N]^k.$$

With respect to z, $L(t, z)$ is a polynomial of degree k, whose first coefficient equals one. As a result of (6), one of the zeros of this polynomial, say $z_1 = z_1(t)$, has the value y_1 when $t = 1$. It is also known that the function $z_1 = z_1(t)$ is a continuous function of t if $0 \leqslant t \leqslant 1$. Then either the curve $z_1 = z_1(t)$ will cut one of the sides of the angle U, when $t = t_0$ and $0 < t_0 < 1$

or the point $z_1 = z_1(0)$ will remain within the angle U. In the first case, Eq. (7) holds for the points:

$$y_1' = z_1(t_0), \qquad y_2' = y_2, \qquad y_m' = t_0 y_m, \qquad m = 3, 4,..., N.$$

It is clear that when $t_0 > 0$ and for the fixed $m = 3, 4,..., N$, the points y_m and y_m' are situated on the same ray coming out of the origin, therefore these points are either both on the sides of the angle U, or both within U. That is, as before, only p of the points $y_2', y_3',..., y_N'$ are situated within the angle U. Furthermore, y_1' is situated on a side of the angle U, therefore only p out of the points $y_1', y_2',..., y_N'$ are situated within the angle U and the proof is established by induction.

In the second case,

$$L(0, z(0)) = [z_1(0), y_2, 0, 0,..., 0]^k = 0.$$

Then the Eq. (7) holds for $y_1' = z_1(0)$ and $y_2' = y_2$ which are inside U and the lemma is proved in the second case also.

Remark. It is possible to make Lemma 2 more exact. If at least one of the points $y_1, y_2,..., y_N$ is situated within the angle U then all of the points $y_1', y_2',..., y_s'$ as defined above, can be considered to be situated within the angle U. In fact, let us assume that y_1 is situated within the angle U. Let us investigate the equation

$$[z, y_2 \exp(ix_2 t), y_3 \exp(ix_3 t),..., y_N \exp(ix_N t)]^k = 0,$$

where $x_m = 0$ if the point y_m is found within the angle U; $x_m = 1$ if y_m is situated on the side $\arg z = 0$; and $x_m = -1$ if y_m is on the side $\arg z = \varphi$ of the angle U. One of the roots of this equation, say $z_1 = z_1(t)$, takes on the value y_1, when $t = 0$, and is also continuous function of t. Therefore the number $t_0 > 0$ can be taken small enough such that all of the points $y_1'' = z_1(t_0)$, $y_m'' = y_m \exp(ix_m t_0)$, $m = 2, 3,..., N$ will fall within the angle U. Now it is sufficient to apply Lemma 2 for the points y_m''.

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Let s points be situated on two rays which come out of the origin of the complex plane. Let us denote the points on one of the rays by $a_j = q_j \exp(i\alpha)$, $j = 1, 2,..., s_1$ and on the other by $b_j = r_j \exp(i(\alpha + \varphi))$, $j = 1, 2,..., s_2$. Here $s_1 + s_2 = s$ and q_j, r_j are absolute values of the complex numbers a_j and b_j. Then the homogeneous expression (3) of power k from the complex numbers $a_1, a_2, \ldots, a_{s_1}; b_1, b_2, \ldots, b_{s_2}$ can be written in the form

$$[a_1, a_2, \ldots, a_{s_1}; b_1, b_2, \ldots, b_{s_2}]^k = e^{ik\alpha} \sum_{m=0}^{s} C_m e^{im\varphi}, \tag{8}$$

where

$$C_m = A_m B_{k-m},$$

$$A_m = [q_1, q_2, \ldots, q_{s_1}]^m, \qquad B_m = [r_1, r_2, \ldots, r_{s_2}]^m, \qquad m > 0$$

and

$$A_0 = B_0 = 1.$$

We now prove the following lemma for the quantities A_m.

LEMMA 3. *If $q_1, q_2, \ldots, q_{s_1}$ are positive numbers, then for each integral m,*

$$R_m = (A_m)^2 - A_{m-1}A_{m+1} > 0$$

when $s_1 > 1$, and

$$R_m = 0$$

when $s_1 = 1$.

Proof. When $s_1 = 1$, $A_m = q_1{}^m$, and therefore

$$R_m = q_1^{2m} - q_1^{m-1} q_1^{m+1} = 0.$$

Now let us assume that lemma is proved for $s_1 = p$ and for each integral m,

$$R_m = (A_m)^2 - A_{m-1}A_{m+1} \geqslant 0 \tag{9}$$

and designate

$$E_m = [q_1, q_2, \ldots, q_p, q]^m,$$

where $q > 0$.

It is easy to see that

$$E_m = E_{m-1}q + A_m, \tag{10}$$

therefore

$$E_m = A_m + qA_{m-1} + \cdots + q^{m-1}A_1 + q^m. \tag{11}$$

From (9) we get the following inequalities:

$$D_{m,i} = A_m A_{m-i+1} - A_{m+1}A_{m-i} \geqslant 0, \qquad i = 1, 2, \ldots, m,$$

in fact

$$D_{m,i} = A_m A_{m-i} \left(\frac{A_{m-i+1}}{A_{m-i}} - \frac{A_{m+1}}{A_m} \right)$$

$$= A_m A_{m-i} \left(\frac{A_{m-i+1}}{A_{m-i}} - \frac{A_{m-i+2}}{A_{m-i+1}} + \frac{A_{m-i+2}}{A_{m-i+1}} - \cdots - \frac{A_m}{A_{m-1}} + \frac{A_m}{A_{m-1}} - \frac{A_{m-1}}{A_m} \right)$$

$$= A_m A_{m-i} \left(\frac{R_{m-i+1}}{A_{m-i} A_{m-i+1}} + \frac{R_{m-i+2}}{A_{m-i+1} A_{m-i+2}} + \cdots + \frac{R_{m-1}}{A_{m-2} A_{m-1}} + \frac{R_m}{A_{m-1} A_m} \right) \geqslant 0.$$

Then from (10) and (11)

$$\begin{aligned} (E_m)^2 - E_{m+1} E_{m-1} &= (A_m + E_{m-1} q) E_m - (A_{m+1} + E_m q) E_{m-1} \\ &= A_m E_m - A_{m+1} E_{m-1} \\ &= A_m (A_m + A_{m-1} q + \cdots + A_1 q^{m-1} + q^m) - \\ &\quad - A_{m+1} (A_{m-1} + A_{m-2} q + \cdots + A_1 q^{m-2} + q^{m-1}) \\ &= D_{m,1} + D_{m,2} q + \cdots + D_{m,m} q^{m-1} + A_m q^m > 0, \end{aligned}$$

thus the lemma is proved.

A similar inequality holds for B_m :

$$R_m{}' = (B_m)^2 - B_{m+1} B_{m-1} > 0 \tag{12}$$

when $s_2 > 1$, and $R_m{}' = 0$ when $s_2 = 1$.

LEMMA 4. *Let us define* s_1, s_2, q_j, r_j, *and* C_m *as above. Now if* $s_1 = s_2 = 1$ *and* $r_1 = q_1$, *then all the* C_m *are equal. In all other cases there exists a number* p, $0 \leqslant p \leqslant k$ *such that* $C_0 < C_1 < \cdots < C_p$ *and* $C_{p+1} > C_{p+2} > \cdots > C_k$.

Proof. Let us investigate the difference

$$C_{m+1} - C_m = A_{m+1} B_{k-m-1} - A_m B_{k-m} = A_{m+1} B_{k-m} \left(\frac{B_{k-m-1}}{B_{k-m}} - \frac{A_m}{A_{m+1}} \right).$$

As a result of (12), when m increases, the ratio B_{k-m-1}/B_{k-m} decreases if $s_2 > 1$, because

$$\frac{B_{k-m-2}}{B_{k-m-1}} - \frac{B_{k-m-1}}{B_{k-m}} = - \frac{R'_{k-m-1}}{B_{k-m-1} B_{k-m}} < 0.$$

The ratio $-A_m/A_{m+1}$ also decreases with increasing m if $s_1 > 1$, since

$$- \frac{A_{m+1}}{A_{m+2}} + \frac{A_m}{A_{m+1}} = - \frac{R_{m+1}}{A_{m+2} A_{m+1}} < 0.$$

Therefore if at least one of the numbers s_1 or s_2 is greater than one, then $[(B_{k-m-1}/B_{k-m}) - (A_m/A_{m+1})]$ decreases with increasing m. As a result of this, $C_{m+1} - C_m$ can change sign no more than once, i.e., there exist p, $0 \leqslant p \leqslant k$ such that $C_1 < C_2 < \cdots < C_p$ and $C_{p+1} > C_{p+2} > \cdots > C_k$.

If $s_1 = s_2 = 1$ then

$$C_{m+1} - C_m = q_1^{m+1} r_1^{k-m} \left(\frac{r_1^{k-m-1}}{r_1^{k-m}} - \frac{q_1^{\ m}}{q_1^{m+1}}\right) = q_1^{m+1} r_1^{k-m} \left(\frac{1}{r_1} - \frac{1}{q_1}\right).$$

From the last equation, we can easily see that if $r_1 > q_1$ ($r_1 < q_1$) the function C_m decreases (increases) with increasing m. Therefore there exists p ($p = 0$ or k) as is stated in the lemma. If $r_1 = q_1$ then $C_1 = C_2 = \cdots = C_k$.

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LEMMA 5. *Let a_0, a_1,..., a_k be real positive numbers and there exist an p, $0 \leqslant p \leqslant k$, such that*

$$a_0 \leqslant a_1 \leqslant \cdots \leqslant a_p \tag{13}$$

and

$$a_{p+1} \geqslant a_{p+2} \geqslant \cdots \geqslant a_k . \tag{14}$$

Then for the polynomial

$$Q(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0 \tag{15}$$

there are no zeros on the arc

$$| z | = 1, \qquad -2\pi/(k+1) < \arg z < 2\pi/(k+1). \tag{16}$$

The polynomial $Q(z)$ has zeros at $z = \exp(\pm 2\pi i/(k+1))$ only when $a_0 = a_1 = \cdots = a_k$.

Proof. It is sufficient to prove that when there are at least two unequal numbers among the numbers a_i, then the polynomial $Q(z)$ has no zeros on the arc

$$| z | = 1, \qquad -2\pi/(k+1) \leqslant \arg z \leqslant 2\pi/(k+1).$$

In this case we can assume that in at least one of the collections of numbers (13) or (14) there are at least two different numbers.

In fact, let us assume that $a_0 = a_1 = \cdots = a_p = a$ and $a_{p+1} = a_{p+2} = \cdots = a_k = b$, where $b \neq a$. Without loss of generality take $b > a$. Then it is possible to transfer a_{p+1} from (14) to (13) without changing the conditions of the lemma. Thus in (13) not all numbers are equal. Therefore for purposes of the proof we will assume that in the collection (13) not all of the numbers are equal, that is

$$a_0 < a_p . \tag{17}$$

All of the coefficients of the polynomial (15) are real, therefore it is sufficient to investigate the arc

$$|z| = 1, \qquad 0 \leqslant \arg z \leqslant 2\pi/(k+1).$$

It is clear that $z = 1$ is not a zero of (15). We now look at $z = \exp(i\varphi)$ where

$$0 \leqslant \varphi \leqslant 2\pi/(k+1) \tag{18}$$

and define the following vectors in the complex plane

$$A_s = a_s \exp(is\varphi), \qquad s = 0, 1,..., k.$$

Let us assume that u_1 and u_2 are the angles between A_0 and A_p, and between A_{p+1} and A_k, respectively, and assume that b_1 and b_2 are the bisectors of u_1 and u_2,

$$b_1\colon \arg z = p\varphi/2; \qquad b_2\colon \arg z = ((p+1)+k)\varphi/2.$$

As a result of (18), the angle (counterclockwise) between b_1 and b_2 is not greater than π (and is equal to π only if $\varphi = 2\pi/(k+1)$).

Therefore, the bisector b_1 and the vectors A_p and A_{p+1} are on the same side of the line L which passes through the bisector b_2.

It is easy to see that at least one of the angles u_1 and u_2, say u_1 is smaller than π. As a result of (13) and (17) the vectors $A_0 + A_p$, $A_1 + A_{p-1}$, $A_2 + A_{p-2}$,... all fall in the angle γ between b_1 and A_p, and the first of them falls inside γ.

Therefore the vector

$$S_1 = A_0 + A_1 + \cdots + A_p$$

is not equal to zero (since $\gamma \leqslant \pi$) and falls within γ. From (14) we get that the vector

$$S_2 = A_{p+1} + A_{p+2} + \cdots + A_k$$

is situated on the same side of the line L (which passes through the bisector b_2) as the vector A_{p+1}. S_2 can be also on the line L.

As a result of this we conclude that the vectors S_1 and S_2 are on the same side of the line L, and only the vector S_2 may be situated on this line. Therefore $S_1 + S_2 \neq 0$ if z is on the arc (18).

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Proof of the Theorem. The theorem is clearly true for $k = 0$, and at the beginning of this paper we showed that it holds for $k = 1$. Let us assume that the zeros of the polynomial $P(z)$: $z_1, z_2, \ldots, z_{n-k+1}$ fall within the angle $U(U: 0 \leqslant \arg z \leqslant 2\pi/(k+1))$ and of them $z_1, z_2, \ldots, z_p$ are not zero. $p \geqslant 2$ since

$$| a_0 | + | a_1 | + \cdots + | a_{n-k+1} | \neq 0.$$

From Lemma 1

$$[z_1, z_2, \ldots, z_{n-k+1}]^k = [z_1, z_2, \ldots, z_p]^k = 0. \tag{19}$$

Clearly (19) does not hold if all of z_i, $i = 1, 2, \ldots, p$ are on the same ray coming out of the origin. Therefore there are two possibilities

(a) At least one of z_i, $i = 1, 2, \ldots, p$, e.g., z_1 is situated within U.

(b) All of the z_i, $i = 1, 2, \ldots, p$ are situated on the sides of the angle U.

In the first case, as a result of the Remark to Lemma 2 there are always points $y_1, y_2, \ldots, y_s$, $1 \leqslant s \leqslant p$ situated on two rays L_1 and L_2 which originate at the origin of the complex plane and fall within U and

$$[y_1, y_2, \ldots, y_s]^k = 0. \tag{19'}$$

As in the beginning of the part III we can write the last equation as

$$\sum_{m=0}^{k} C_m z^m = 0, \tag{20}$$

where $z = \exp(i\varphi)$ and $0 \leqslant \varphi < 2\pi/(k+1)$ being the angle between the rays L_1 and L_2.

From Lemma 3 it turns out that for the coefficients C_m of the polynomial (20), Lemma 4 holds. But $\varphi < 2\pi/(k+1)$ and therefore according to Lemma 5, Eq. (20) can not hold.

In the second case all the points are situated on the rays $\arg z = 0$ and $\arg z = 2\pi/(k+1)$, therefore it is possible to write the Eq. (19') in form (20) where as in the first case for C_m Lemma 4 holds and $z = \exp(i\varphi)$ $(\varphi = 2\pi/(k+1))$. From Lemma 5 in this case we conclude that equation (20) holds only if $C_0 = C_1 = \cdots = C_k$ and this exists only if $p = 2$, $z_1 = r$ and $z_2 = r\exp(2\pi/(k+1))$. Then

$$P(z) = z^{n-k-1}(z^{k+1} + a), \qquad a \neq 0.$$

This completes the proof of the theorem.

References

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